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The Draftsman's and Related Equations

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1. INTRODUCTION

A standard design problem consists in interpolating a smooth surface through two orthogonal families of intersecting parallel plane curves. In suitable Cartesian coordinates, this is equivalent to the following: One is given plane sections

$$Z(x_i, y) = g_i(y) \text{ and } Z(x, y_j) = f_j(x) \quad (x_0 < x_1 < \ldots < x_M; y_0 < y_1 < \ldots < y_N) \tag{1}$$

subject to the compatibility conditions

$$g_i(y_j) = f_j(x_i) = Z(x_i, y_j). \tag{1'}$$

The problem is to construct a "smooth" bivariate function $Z(x,y)$ defined for $(x,y) \in \mathscr{R} = [x_0, x_M] \times [y_0, y_N]$ which interpolates through this curve network.

The purpose of this paper is to describe and analyze explicit *local* schemes for solving this problem by interpolating surfaces $z = Z(x,y)$ which satisfy, for a specified positive integer p, the hyperbolic differential equation (DE)

$$\frac{\partial^{4p}}{\partial x^{2p}\, \partial y^{2p}} Z(x,y) \equiv Z^{(2p,2p)}(x,y) = 0. \tag{2}$$

By "local", we mean that (for given p) the scheme applies independently on each subrectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ of $\mathscr{R}$. That is, we treat in this paper the special case $M = N = 1$ of (1)-(1′).

When the $g_i(y)$ and $f_j(x)$ are piecewise polynomial functions of degree (at most) $2p - 1$ the schemes treated below include bivariate *Hermite* interpolation [*4*]. For $p = 1, 2$ and general smooth $g_i(y)$ and $f_j(x)$, they reduce to standard drafting techniques. For larger p, they give a rigorous interpretation of the "blending function" methods described by Coons [*5*] in a widely read but unpublished paper[1].

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[1] In [*1*, pp. 262–264], Coons' work is briefly described.

The novelty of the present paper consists in its correlation of practical techniques for interpolation with boundary value problems for the DE (2) and, more generally, with the *well-set boundary value problem* defined by the inhomogeneous DE

$$Z^{(2p,2p)}(x,y) = s(x,y), \tag{3}$$

and suitable "compatibility conditions" (5) at the corners.

In a related investigation [*8*], the local methods discussed here are generalized to global schemes for bivariate interpolation. Instead of local Hermite polynomials, the global schemes make use of spline functions to blend together the curves comprising the given network. Within each subrectangle the spline-weighted interpolant $Z(x,y)$ depends on all the functional information given over the complete domain $\mathscr{R}$, not just that on ∂R_{ij}. The global, spline-weighted schemes of [*8*] bear the same relationship to the local methods described herein as the tensor product spline schemes (e.g., bicubic splines [*7*]) bear to bivariate Hermite interpolation [*4*]. Tensor product schemes, which are a special case of these new schemes, are associated with finite dimensional linear function spaces. The methods described here and in [*8*] are not.

2. Uniqueness Theorem

Since the present paper discusses only the case $M = N = 1$ of (1)-(1′), we can assume that $\mathscr{R}$ is the unit square $S = [0,1] \times [0,1]$, into which any R_{ij} can be transformed affinely by $x \mapsto \alpha x + \alpha'$, $y \mapsto \beta y + \beta'$. This affine transformation takes $Z^{(2p,2p)}$ into $(\alpha\beta)^{-2p} Z^{(2p,2p)}$. We therefore consider the inhomogeneous DE (3) on the unit square with associated boundary conditions

$$Z^{(l,0)}(i,y) = g_i{}^l(y), \qquad Z^{(0,l)}(x,j) = f_j{}^l(x) \tag{4}$$

for $i, j = 0, 1$ and $l = 0, 1, \ldots, p-1$. Our first result is that Eqs. (3) and (4) define a well-set boundary value problem, provided that the boundary conditions are *compatible*:

$$\frac{d^{l'}}{dy^{l'}}[g_i{}^l(y)]\bigg|_{y=y_j} = \frac{d^l}{dx^l}[f_j{}^{l'}(x)]\bigg|_{x=x_i} = Z^{(l,l')}(i,j) \tag{5}$$

for $i, j = 0, 1$ and $l, l' = 0, 1, \ldots, p-1$. We first prove uniqueness.

Theorem 1. *If* $Z^{(2p,2p)}(x,y) = 0$ *in* S *with* $Z^{(l,0)}(i,y) = Z^{(0,l)}(x,j) = 0$ *for* $i, j = 0, 1$ *and* $l = 0, 1, \ldots, p-1$, *then* $Z(x,y) \equiv 0$ *in* S.

Proof. Consider $U(x,y) = Z^{(0,2p)}(x,y)$. Since $U^{(2p,0)}(x,y) = 0$, we have

$$U(x,y) = a_0(y) + a_1(y)x + \ldots + a_{2p-1}(y)x^{2p-1}. \tag{6}$$

On the other hand, the boundary conditions $Z^{(l,0)}(i,y) \equiv 0$ for $i = 0$, 1 and $l = 0, 1, \ldots, p-1$, imply

$$0 \equiv Z^{(l,2p)}(i,y) = U^{(l,0)}(i,y), \qquad i = 0, 1 \text{ and } l = 0, 1, \ldots, p-1. \tag{7}$$

By the uniqueness of Hermite interpolation of degree $(2p-1)$ in x [*6*, p. 37] conditions (6)–(7) imply $a_0(y) = a_1(y) = \ldots = a_{2p-1}(y) = 0$ for each fixed $y \in [0,1]$ and so:

$$0 \equiv U(x,y) = Z^{(0,2p)}(x,y). \tag{8}$$

In turn, this implies

$$Z(x,y) = b_0(x) + b_1(x)y + \ldots + b_{2p-1}(x)y^{2p-1}. \tag{9}$$

We now use the boundary conditions $Z^{(0,l)}(x,j) = 0$ for $j = 0$, 1 and $l = 0$, 1, $\ldots, p-1$. By the uniqueness of Hermite interpolation of degree $(2p-1)$ in y, the preceding boundary conditions imply, for each fixed $x \in [0,1]$, that $b_0(x) = b_1(x) = \ldots = b_{2p-1}(x) = 0$. Therefore, $Z(x,y) \equiv 0$, as claimed.

COROLLARY. *For any positive integer p and any $s(x,y) \in C^{0,0}[S] = C[S]$, the hyperbolic equation*

$$Z^{(2p,2p)}(x,y) = s(x,y) \text{ in } S, \tag{10}$$

with the boundary conditions

$$Z^{(l,0)}(i,y) = g_i^{\,l}(y), \qquad Z^{(0,l)}(x,j) = f_j^{\,l}(x) \qquad \text{for } i,j = 0, 1 \text{ and } l = 0, 1, \ldots, p-1 \tag{11}$$

has at most one solution.

3. GREEN'S FUNCTION FOR $\partial^{4p}/\partial x^{2p}\partial y^{2p}$

We now prove an existence theorem for the hyperbolic DE (10) in the special case that all $g_i^{\,l}(y)$ and $f_j^{\,l}(x)$ are zero. To this end, we let $G_{2p}(x;\xi)$ be the *Green's function* for the ordinary differential equation $u^{(2p)}(x) = \varphi(x)$, $u^{(l)}(0) = u^{(l)}(1) = 0$ for $l = 0$, 1, $\ldots$, $p-1$. These Green's functions are well known [*2*]; for example for $p = 2$:

$$12G_4(x;\xi) = |x-\xi|^3 - (x+\xi)^3 + 6x\xi(x+\xi)(1+x\xi) - 4x^2\xi^2(3+x\xi).$$

THEOREM 2. *Let $s(x,y) \in C[S]$. Then the function*

$$Z(s,y) = \iint_S G_{2p}(x;\xi)\, G_{2p}(y;\eta)\, s(\xi;\eta)\, d\xi\, d\eta \tag{12}$$

belongs to $C^{(2p,2p)}[S]$, and it satisfies (10) *and*

$$Z^{(l,0)}(i,y) = Z^{(0,l)}(x,j) = 0. \qquad 0 \leqq x, y \leqq 1, \tag{13}$$

for $i, j = 0$, 1 and $l = 0, 1, \ldots, p-1$.

Proof. The proof is a repeated application of Leibniz' formula for differentiating under the integral sign with respect to x and y. For $l, l' = 0, 1, 2, \ldots, 2p - 1$, we have

$$Z^{(l,l')}(x,y) = \int_0^1 d\xi \int_0^1 d\eta \, G_{2p}^{(l,0)}(x;\xi)\, G_{2p}^{(0,l')}(y;\eta)\, s(\xi,\eta).$$

For $l, l' = 2p$, only a little more care is required [3, p. 278].

By the Corollary of Theorem 1, at most one function Z can satisfy conditions (10) and (13). Therefore, Theorem 2 shows that

$$G_{2p}(x;\xi)\, G_{2p}(y;\eta) = \Gamma_{2p}(x,y;\xi,\eta)$$

is the *Green's function* for the differential operator $\partial^{4p}/\partial x^{2p}\,\partial y^{2p}$ and the boundary conditions (13) in the unit square S.

4. Mangeron's Theorem

We now prove Mangeron's Theorem [9]. This is a straightforward consequence of the uniqueness theorems above and standard formulas for drafting (blending with linear weighting functions).

Theorem 3. *The fourth-order draftsman's equation*

$$Z_{xxyy}(x,y) \equiv Z^{(2,2)}(x,y) = 0 \quad \text{in } S \tag{14}$$

with compatible boundary conditions on ∂S:

$$Z(x,j) = f_j(x); \qquad Z(i,y) = g_i(y) \qquad (i,j = 0,1) \tag{15}$$

where $f_j(x)$, $g_i(y) \in C^2[0,1]$ *and*

$$f_j(i) = g_i(j) = Z_{ij}, \tag{15'}$$

has the (unique) solution $Z \in C^{2,2}[S]$

$$\begin{aligned} Z(x,y) = {} & (1-y)f_0(x) + yf_1(x) + (1-x)g_0(y) + xg_1(y) \\ & - (1-x)(1-y)Z_{00} - (1-x)yZ_{01} - x(1-y)Z_{10} - xyZ_{11}. \end{aligned} \tag{16}$$

Proof. Since each summand $u(x,y)$ on the right side of (16) is linear in x or in y, either $u^{(2,0)} = 0$ or $u^{(0,2)} = 0$. In either case, $u^{(2,2)} = 0$. By direct substitution in (16) we have, along the edge $y = 0$,

$$Z(x,0) = f_0(x) + (1-x)g_0(0) + xg_1(0) - (1-x)Z_{00} - xZ_{10} = f_0(x),$$

and likewise for the other three edges of S. It is easy to verify that $Z \in C^{2,2}[S]$.

Theorem 2 (for $p = 1$) and Theorem 3 show that the fourth-order hyperbolic operator $D^{(2,2)} = \partial^4/\partial x^2\,\partial y^2$ in any coordinate rectangle leads to a well-set boundary value problem surprisingly like the Dirichlet problem for the (elliptic) Laplace operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We now give two examples which bring out some basic differences between these problems.

Example 1. Consider the boundary conditions $f_i(x) = 4x - 4x^2$ $(i = 0, 1)$; $g_j(y) = 4y - 4y^2$ $(j = 0, 1)$. These functions assume their minimum value (zero) at the endpoints of the interval $[0, 1]$, and their maximum value (one) when $x = 1/2$ and $y = 1/2$. Substituting into (16), we get the interpolating function

$$Z(x, y) = 4(x - x^2) + 4(y - y^2),$$

whose maximum value is two.

This shows that the Maximum Principle does *not* hold for the interpolation formula (16). On the other hand, it is true that

$$\mathrm{Max}_S |Z(x, y)| \leqslant 2\, \mathrm{Max}_{\partial S} |Z(x, y)|. \tag{17}$$

Hence we do have an analog of Harnack's First Convergence Theorem for harmonic functions (and of Weierstrass' Theorem for analytic complex functions).

THEOREM 4. *If the boundary values of a sequence of solutions of the draftsman's equation* (14) *converge uniformly, then the interior values also converge uniformly, and to a solution of the draftsman's equation.*

On the other hand, mean square convergence of the boundary values does *not* imply mean square convergence for interior values, as the following modification of Example 1 shows.

Example 2. Consider the boundary conditions

$$f_i(x) = (4x - 4x^2)^{1/n}, \qquad g_j(y) = (4y - 4y^2)^{1/n}.$$

As $n \to \infty$, these boundary values converge in every L_p-norm

$$\|(f, g)\|_p = \left[\int_{\partial S} |Z(x, y)|^p\, ds\right]^{1/p}, \qquad 1 \leqslant p < +\infty,$$

to the boundary value $Z \equiv 1$. However, the interior values given by (13),

$$Z_n(x, y) = (4x - 4x^2)^{1/n} + (4y - 4y^2)^{1/n}$$

converge to the value $Z(x, y) \equiv 2$ in L_p-norm $[\iint_S |Z(x, y)|^p\, dx\, dy]^{1/p} = \|Z\|_p$.

To yield a well-set boundary value problem in an L_p-norm, one should set

$$\|(f, g)\|_p = \left[\int_{\partial S} |Z(x, y)|^p\, ds + \sum_{i, j = 0} |Z(i, j)|^p\right]^{1/p}.$$

That is, one should assign a weight of one to corner points.

5. Extension to $D^{(4,4)}$

We now extend Mangeron's results to the eighth-order hyperbolic operator $D^{(4,4)} = \partial^8/\partial x^4\,\partial y^4$, which is related to the biharmonic operator

$$\nabla^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$$

in much the same way that $D^{(2,2)}$ is related to ∇^2. We do this because the methods used to make the extension to this case, with $p = 2$, can be combined with induction to treat the case of general p. Also, since the cases $p = 1$ and $p = 2$ are probably the most important computationally, the explicit formulas are worth presenting.

We first show that the operator $D^{(4,4)}$ defines a well-set boundary value problem for almost exactly the same boundary conditions as the usual ones for ∇^4.

Lemma 1. *The partial differential equation*

$$Z^{(4,4)}(x,y) = 0 \quad \textit{in } S, \tag{18}$$

with boundary conditions

$$Z(x,j) = Z(i,y) = 0, \tag{18a}$$

$$Z_y(x,j) = f_j^{\,1}(x), \qquad Z_x(i,y) = g_i^{\,1}(y) \quad \textit{on } \partial S \tag{18b}$$

where $f_j^{\,1}(x)$ and $g_i^{\,1}(y) \in C^4[0,1]$ and satisfy the compatibility relations

$$Z_{xy}(i,j) = \frac{d}{dx} f_j^{\,1}(i) = \frac{d}{dy} g_i^{\,1}(j) \quad \textit{for } i,j = 0,\ 1,$$

has the unique class $C^{4,4}[S]$ solution

$$\begin{aligned} Z(x,y) = {} & w_0(y) f_0^{\,1}(x) + w_1(y) f_1^{\,1}(x) + w_0(x) g_0^{\,1}(y) + w_1(x) g_1^{\,1}(y) \\ & - w_0(x)\, w_0(y) Z^{(1,1)}(0,0) - w_0(x)\, w_1(y) Z^{(1,1)}(0,1) \\ & - w_1(x)\, w_0(y) Z^{(1,1)}(1,0) - w_1(x)\, w_1(y) Z^{(1,1)}(1,1) \end{aligned} \tag{19}$$

where the weighting ("blending", see [5]) functions w_0 and w_1 are

$$w_0(x) = (x-1)^2 x \tag{19a}$$

$$w_1(x) \equiv w_0(1-x) = x^2(1-x). \tag{19b}$$

Proof. By inspection we see that $Z(x,y)$ does, in fact, vanish on ∂S and that its normal derivatives along ∂S interpolate to the given values, $f_j^{\,1}(x)$ and $g_i^{\,1}(y)$ $(i,j = 0,1)$. Also, since the functions w_0 and w_1 are cubic polynomials, $Z(x,y)$ satisfies $Z^{(4,4)} = 0$ in S for any compatible boundary values. This establishes the existence of a solution in $C^{4,4}[S]$. Uniqueness follows from Theorem 1.

Now, let $Z_1(x,y)$ be the function in (16) and define $Z_3(x,y)$ as the function obtained by taking $p=2$ in (12). Related to the given normal derivative functions $f_j{}^1(x)$ and $g_i{}^1(y)$, define the derived functions

$$\tilde{f}_j{}^1(x) = f_j{}^1(x) - \frac{\partial}{\partial y} Z_1(x,j) \qquad (j = 0, 1)$$
$$\tilde{g}_i{}^1(y) = g_i{}^1(y) - \frac{\partial}{\partial x} Z_1(i,y) \qquad (i = 0, 1). \tag{20}$$

Define $Z_2(x,y)$ to be the function given by (19) if the $f_j{}^1(x)$ and $g_i{}^1(y)\,(i,j = 0, 1)$ are replaced by $\tilde{f}_j{}^1(x)$ and $\tilde{g}_i{}^1(y)$, respectively. Then, by combining the results of Theorems 1–3 and Lemma 1, we obtain the following results.

THEOREM 5. *The hyperbolic* DE (18) *has one and only one solution for which Z and $\partial Z/\partial n$ assume given compatible boundary values of class $C^4[\partial S]$.*

Proof. In our notation, the boundary conditions are

$$Z^{(0,l)}(x,j) = f_j{}^l(x), \qquad Z^{(l,0)}(i,y) = g_i{}^l(y), \tag{21}$$

where $i, j, l = 0, 1$. By "compatible", we mean that

$$Z(i,j) = f_j{}^0(i) = g_i{}^0(j), \qquad Z_{xy}(i,j) = \frac{d}{dx} f_j{}^1(i) = \frac{d}{dy} g_i{}^1(j). \tag{21'}$$

Since we proved uniqueness in Theorem 1, it suffices to prove existence. We shall exhibit the solution as

$$Z(x,y) = Z_1(x,y) + Z_2(x,y). \tag{22}$$

Since the function $Z_1(x,y)$ of (16) satisfies $Z_1^{(2,2)}(x,y) = 0$ it certainly satisfies $Z_1^{(4,4)}(x,y) = 0$; and it interpolates to the given values of $Z(x,y)$ on ∂S. However, $Z_1(x,y)$ will not generally satisfy conditions (21) on the normal derivatives of $Z(x,y)$. Expressions (20) represent the difference between the given boundary conditions on $\partial Z/\partial n$ and the first normal derivative of $Z_1(x,y)$ on ∂S. Clearly, the sum $Z_1(x,y) + Z_2(x,y)$ satisfies the homogeneous differential equation and the boundary conditions on Z and $\partial Z/\partial n$. Hence, the proof is complete.

COROLLARY 1. *The eighth-order hyperbolic* DE

$$Z^{(4,4)}(x,y) = s(x,y), \qquad \text{with } s \in C^{0,0}[S] \tag{23}$$

has one and only one solution for which Z and $\partial Z/\partial n$ assume given compatible boundary values of class $C^4[\partial S]$. This solution is given explicitly by

$$Z = Z_1 + Z_2 + Z_3, \tag{24}$$

where Z_1 and Z_2 are as in (22) *and Z_3 is given by* (12) *with $p = 2$.*

COROLLARY 2. *Any function* $Z \in C^{4,4}[S]$ *has a decomposition of the form*

$$\begin{aligned} Z(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} x^i y^i + \sum_{j=0}^{3} \varphi_j(x) y^j + \sum_{i=0}^{3} \psi_i(y) x^i \\ + \int_0^1 \int_0^1 \Gamma_4(x,y;\,\xi,\eta) s(\xi,\eta)\, d\xi\, d\eta \end{aligned} \tag{25}$$

where the φ_j *and* ψ_i *are in* $C^4[0,1]$, $s \in C[S]$, *and* Γ_4 *is the Green's function of Section* 3.

6. Higher Order Equations

An obvious corollary to the preceding results is that any function $Z(x,y) \in C^{(4,4)}[S]$ admits an expansion of the form (24). For functions of one variable, Taylor's series (or Newton's interpolation formula with remainder) is an expansion of the function in terms of its value and derivatives at a single point. In contrast, the Hermite expansion of a univariate function involves the values of the function and its derivatives at the two endpoints of the unit interval [*6*, p. 37]. For bivariate functions, Taylor's series is an expansion about a single point; whereas the expansion of $Z(x,y)$ in Theorem 6 below is an expansion of the function in terms of its value and the values of its first $p-1$ normal derivatives on the perimeter of the unit square.

We have already noted the analogy between the elliptic operators ∇^2 and ∇^4 and the hyperbolic operators $(\partial^2/\partial x\,\partial y)^2$ and $(\partial^2/\partial x\,\partial y)^4$. Still more generally, the operator $(\partial^2/\partial x\,\partial y)^{2p}$ is associated with a well-set boundary value problem for which the value of the function $Z(x,y)$ and its first $p-1$ normal derivatives are specified on the boundary of S. The weighting functions, analogous to those in expressions (16) and (19), which must be used to satisfy the boundary conditions on each lth normal derivative of $Z(x,y)$, are the cardinal functions for Hermite interpolation by polynomials of degree $2l-1$ (cf. [*6*, p. 37]). Therefore, the operator $\partial^{4p}/\partial x^{2p}\partial y^{2p}$ has polynomial weighting functions of degree $2p-1$ or less. These observations are formalized in the following theorem, due essentially to Coons [*5*]; the w_i^n are the cardinal functions of [*6*, p.37, (2.5.22)].

THEOREM 6. *The hyperbolic* DE (2) *with the compatible boundary conditions*

$$Z^{(l,0)}(x_i,y) = g_i{}^l(y), \qquad Z^{(0,l)}(x,y_j) = f_j{}^l(x), \tag{26}$$

for $i,j = 0, 1$ *and* $l = 0, 1, \ldots, p-1$ *has the solution*

$$Z(x,y) = \sum_{n=1}^{p} Z_n(x,y),$$

where Z_1 *is given by* (16), Z_2 *is as in Theorem* 5 *and recursively,* $Z_{n+1}(x,y)$ *is given by*

$$\begin{aligned}Z_{n+1}(x,y) = {} & w_0{}^n(y)\tilde{f}_0{}^n(x) + w_1{}^n(y)\tilde{f}_1{}^n(x) + w_0{}^n(x)\tilde{g}_0{}^n(y) + w_1{}^n(x)\tilde{g}_1{}^n(y) \\ & - w_0{}^n(x)\,w_0{}^n(y)\tilde{Z}^{(n,n)}(0,0) - w_0{}^n(x)\,w_1{}^n(y)\tilde{Z}^{(n,n)}(0,1) \\ & - w_1{}^n(x)\,w_0{}^n(y)\tilde{Z}^{(n,n)}(1,0) - w_1{}^n(x)\,w_1{}^n(y)\tilde{Z}^{(n,n)}(1,1) \qquad (27)\end{aligned}$$

where

$$\tilde{f}_j{}^n(x) = f_j{}^n(x) - \frac{\partial^n}{\partial y^n}\left\{\sum_{m=1}^{n} Z_m(x,j)\right\} \qquad (j=0,1) \qquad (28a)$$

$$\tilde{g}_i{}^n(y) = g_i{}^n(y) - \frac{\partial^n}{\partial x^n}\left\{\sum_{m=1}^{n} Z_m(i,y)\right\} \qquad (i=0,1) \qquad (28b)$$

$$\tilde{Z}^{(n,n)}(i,j) = \frac{d^n}{dx^n}\tilde{f}_j{}^n(i) = \frac{d^n}{dy^n}\tilde{g}_i{}^n(j) \qquad (i,j=0,1). \qquad (28c)$$

Proof. The proof is accomplished by induction on p and is completely analogous to the proof of Theorem 5.

COROLLARY 1. *The hyperbolic* DE (3) *with compatible boundary conditions* (26), s in $C[S]$ *and all* $g_i{}^l, f_j{}^l$ *in* $C^2;[0,1]$ *has one and only one solution.*

To find this solution, it suffices to add the function $Z(x,y)$ of Theorem 6 (the solution of the reduced equation) to the function $Zp + 1\ (x,y)$ defined by (12).

Relation to Bivariate Hermite Interpolation

In the special case that all $g_i{}^l$ and $f_j{}^l$ are polynomials of degree $2p-1$ or less, Theorem 6 clearly gives a *finite dimensional* scheme of interpolation by functions of the form:

$$U(x,y) = \sum_{m=0}^{2p-1}\sum_{n=0}^{2p-1} a_{mn}\,x^m y^n. \qquad (29)$$

In particular, if the $g_i{}^l$ and $f_j{}^l$ are the $2p-1$ degree polynomials obtained by univariate Hermite interpolation along the four edges of S to the values and derivatives of (30a)–(30c) below, then the Coons scheme of Theorem 6 yields bivariate Hermite interpolation as a special case.

COROLLARY 2. (Bivariate Hermite Interpolation) *A unique function* $U(x,y)$ *of the form* (29) *is determined over* S *by the specification of the* $4p^2$ *values*:

$$U_{ij} = U(x_i, y_j) \qquad (30a)$$

$$\left.\begin{aligned} U_{ij}^{(l,0)} &= U^{(l,0)}(x_i,y_j) \\ U_{ij}^{(0,l)} &= U^{(0,l)}(x_i,y_j)\end{aligned}\right\} \qquad (30b)$$

$$U_{ij}^{(l,l')} = U^{(l,l')}(x_i,y_j) \qquad (i,j=0,1;\ l,l'=0,1,2,\ldots,p-1). \qquad (30c)$$

In conclusion, we consider again the problem (1)-(1′) stated in Section 1 of interpolating a bivariate function $Z(x,y)$ through a given orthogonal network of intersecting plane curves. The rectangular domain $\mathscr{R} = [x_0,x_M] \times [y_0,y_N]$ over which the network is defined is partitioned into $M \cdot N$ subrectangles R_{ij}. By affinely mapping the unit square into R_{ij} and using the transformed versions of the formulas in this paper, one obtains $M \cdot N$ local bivariate functions $Z_{ij}(x,y)$ which interpolate to $g_{i-1}(y)$, $g_i(y)$, $f_{j-1}(x)$ and $f_j(x)$ and to specified normal derivative conditions on the boundary of each R_{ij}. The totality of these $M \cdot N$ locally defined functions provides the "patch surface" solution described by Coons [5] to the problem (1)-(1′). In [8], a global solution to the same network interpolation problem is obtained by using splines as blending functions (weighting functions) instead of the local Hermite-type polynomial blending functions w_i^n; in (27).

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